Tutorial – Propagation of errors

We now need to consider how to combine different measured values, each having uncertainties, in to a final result. This is the subject of the propagation of experimental uncertainties (or errors).

If you feel that the random error, as obtained by applying the following rules, is much smaller than is reasonable, look around for systematic errors and mention them in the final results.

- **Uncertainty of a measurement = Random errors + Systematic errors**

General formula

In the following discussion, measured quantities are represented by letters (Ex. x, y,…) while the uncertainties on the measurements are represented by a delta Δ followed by a letter (Ex. Δx, Δy, …).

If the result R is a function of measurements $x, y, \ldots$ where $R = f(x, y, \ldots)$ the general formula for the propagation of errors is then

$$\Delta R = \sqrt{\left(\frac{\partial R}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial R}{\partial y}\right)^2 \Delta y^2 + \cdots} \quad ,$$

where $\partial R/\partial x$ is the notation for the partial derivative. It is the derivative of R with respect to x while treating all other variables as constants. This formula of error propagation is assuming that all variables (x, y,…) and their uncertainties are completely independent. The next sections present specific applications of the general formula.

Additions & Subtractions

If the result R is obtained from a series of additions and subtractions as

$$R = \pm Ax \pm By \pm \cdots \quad ,$$

where A and B are constants, then $\partial R/\partial x = \pm A$ and $\partial R/\partial y = \pm B$. Consequently, the error on the result R is given by

$$\Delta R = \sqrt{A^2 \Delta x^2 + B^2 \Delta y^2 + \cdots}\,.$$

Example: Calculate $R = 3x + y$ if $x = (11.54 \pm 0.07)\text{cm}$ and $y = (2.1 \pm 0.2)$ cm.

$$R = 3(11.54\text{ cm}) + 2.1\text{ cm} = 36.72\text{ cm}$$

$$\Delta R = \sqrt{3^2(0.07\text{ cm})^2 + (0.2\text{ cm})^2} = 0.29\text{ cm}$$

We round off the uncertainty to one significant digit and the final answer becomes $(36.7 \pm 0.3)\text{cm}$.

Multiplications & Divisions

If the result R is obtained from a series of products such as

$$R = x^A y^B \cdots \quad ,$$

where A and B are constants, then $\partial R/\partial x = A x^{A-1} y^B \cdots = AR/x$ and $\partial R/\partial y = B x^A y^{B-1} \cdots = BR/y$. Consequently, the error on the result R is given by

$$\Delta R = |R| \sqrt{A^2 \frac{\Delta x^2}{x^2} + B^2 \frac{\Delta y^2}{y^2} + \cdots} \quad .$$

Example: Calculate $R = x^2/y^3$ if $x = (11.54 \pm 0.07)\text{cm}$ and $y = (2.1 \pm 0.2)$ cm.

$$R = (11.54 \text{ cm})^2 (2.1 \text{ cm})^{-3} = 14.38 \text{ cm}^{-1}$$

$$\Delta R = (14.38 \text{ cm}^{-1}) \sqrt{2^2 \left(\frac{0.07 \text{ cm}}{11.54 \text{ cm}}\right)^2 + 3^2 \left(\frac{0.2 \text{ cm}}{2.1 \text{ cm}}\right)^2} = 4.11 \text{ cm}^{-1}$$

We round off the uncertainty to one significant digit and the final answer becomes $(14 \pm 4)\text{cm}^{-1}$.

Trigonometric functions

Consider the case where the result R is obtained from a trigonometric function such as

$$R = x \sin\theta \quad .$$

The partial derivatives needed to use the general equation are then $\partial R/\partial x = \sin\theta$ and $\partial R/\partial\theta = x\cos\theta$. Consequently, the error on the result R is given by

$$\Delta R = \sqrt{\sin^2\theta\,\Delta x^2 + x^2\cos^2\theta\,\Delta\theta^2} \quad ,$$

where θ and $\Delta\theta$ should always be in radians. An angle in radians is the angle in degrees times $\pi/180$.

Example: Calculate $R = x\sin\theta$ if $x = (11.54 \pm 0.07)\text{cm}$ and $\theta = (20 \pm 1)°$.

We first convert the angle in radians: $\theta_{rad} = \pi\theta/180 = (0.349066 \pm 0.017453)$.

$$R = (11.54 \text{ cm}) \sin(0.349066) = 3.9469 \text{ cm}$$

$$\Delta R = \sqrt{\sin^2(0.349066)\,(0.07 \text{ cm})^2 + (11.54 \text{ cm})^2 \cos^2(0.349066)\,(0.017453)^2} = 0.190773 \text{ cm}$$

We round off the uncertainty to one significant digit and the final answer becomes $(3.9 \pm 0.2)\text{cm}$.

Other functions

In any other cases ($R = \ln x$, $R = e^x$, …), you should always start from the general equation to derive the error propagation formula as we did for the trigonometric function example above.